1. Let $S: \mathbb{P}_3 \to \mathbb{P}_3$ be defined by

$$S[p(x)] = p''(x) - 2xp'(x)$$

(a) Find the eigenvalues and eigenvectors of S.

(b) Let $\mathcal{E} = \{1, x, x^2, x^3\}$ (the standard basis of $\mathbb{P}_3$). Is S diagonalizable? If so, write $M_S^{\mathcal{E},\mathcal{E}}$ as $M_S^{\mathcal{E},\mathcal{E}} = P^{-1}DP$ where D is diagonal and P is an invertible matrix

2. Let $T: \mathbb{R}^5 \to \mathbb{R}^5$. Its associate matrix with respect to the standard basis is

$$M_T = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 6 & 0 \\ 0 & 0 & -4 & 0 & 12 \\ 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & -16 \end{pmatrix}$$

(a) Without computations, explain why T is diagonalizable.

(b) Diagonalize T.

3. Suppose that $T(x) = A\vec{x}$ $(T: \mathbb{R}^n \to \mathbb{R}^n)$ is diagonalizable. Justify the following statements:

(a) A is invertible if and only if 0 is not an eigenvalue.

(b) If λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n.

(c) If λ is an eigenvalue of A and A is invertible, then λ^{-1} is an eigenvalue of A^{-1}.

4. Let $T: V \to V$ be a linear mapping where $\dim V = 3$ with associated matrix with respect to some basis B

$$M_T^B = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 3 & 0 \\ 14+3a & a & 3 \end{pmatrix}$$

For what values of $a \in \mathbb{R}$ is T diagonalizable?